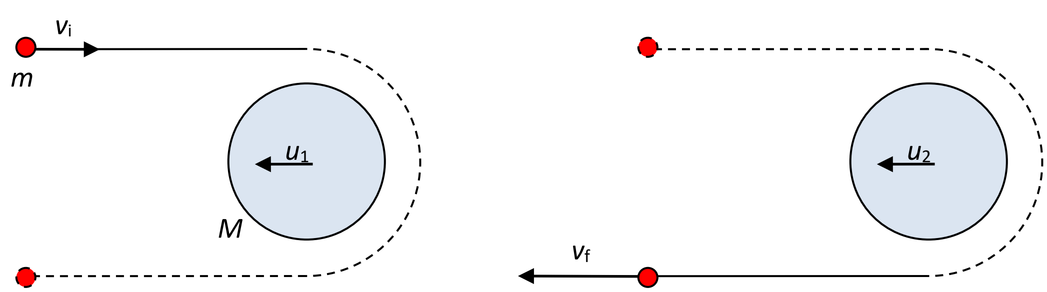
# Teacher notes

# Topic D

## The gravitational slingshot effect

An interesting effect takes place when a satellite is directed towards a planet, swings around the planet and emerges with an increased speed. The effect was used most famously with the Voyager 2 spacecraft that approached Saturn on August 27, 1981 and was redirected with an increased speed towards Uranus where the effect was repeated on January 30, 1986. The spacecraft has since left the solar system. The effect was used many times before and after Voyager.

The diagram shows a simplified and extreme case of the effect in which the satellite (mass *m*) approaches the planet (mass *M*) with a velocity that is opposite to the planet's velocity. In reality the path of the satellite is a hyperbolic path around the planet. The diagram shows the satellite approaching with speed $v_i$ and leaving in the opposite direction with speed $v_f$. The initial speed of the planet is $u_1$ and the final speed is $u_2$. As we might expect, and as will see, $u_1 \approx u_2$. All speeds are relative to the Sun.




We apply the laws of conservation of momentum and energy to the satellite-planet system:

Momentum: $mv_i - Mu_1 = -mv_f - Mu_2$

Kinetic energy: $\frac{1}{2}mv_i^2 + \frac{1}{2}Mu_1^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}Mu_2^2$

(The procedure takes place quickly and so there is no appreciable change in the position of the planet and hence no change in gravitational potential energy. For the same reason, the impulse provided by the gravitational force from the Sun (which is an external force) is very small and so momentum is conserved.)

From the first equation: $u_2 = \dfrac{Mu_1 - mv_f - mv_i}{M}$. Substituting this into the second equation we find

$$\frac{1}{2}mv_i^2 + \frac{1}{2}Mu_1^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}M\left(\frac{Mu_1 - mv_f - mv_i}{M}\right)^2$$

This becomes (terms in color cancel out):

$$\begin{aligned}
&Mmv_i^2 + M^2u_1^2 = Mmv_f^2 + (Mu_1 - mv_f - mv_i)^2 \\
&Mmv_i^2 + M^2u_1^2 = Mmv_f^2 + M^2u_1^2 + m^2v_f^2 + m^2v_i^2 - 2Mmu_1v_f - 2Mmu_1v_i + 2m^2v_fv_i \\
&m(m+M)v_f^2 - 2m(Mu_1 - mv_i) + mv_i\big((m-M)v_i - 2Mu_1\big) = 0
\end{aligned}$$

This is a quadratic equation for the final speed. The discriminant is (we use the simpler form when the $x$ coefficient is even: $ax^2 + 2bx + c = 0$ has solutions $\dfrac{-b \pm \sqrt{b^2 - ac}}{a}$):

$$\Delta = m^2(Mu_1 - mv_i)^2 - m(m+M)mv_i\big((m-M)v_i - 2Mu_1\big)$$

and greatly simplifies:

$$\begin{aligned}
\Delta &= m^2(M^2u_1^2 - 2Mmu_1v_i + m^2v_i^2) - m^2(m^2 - M^2)v_i^2 + 2M(m+M)m^2v_iu_i \\
&= m^2M^2u_1^2 - 2Mm^3u_1v_i + m^4v_i^2 - m^4v_i^2 + m^2M^2v_i^2 + 2Mm^3v_iu_i + 2M^2m^2v_iu_i \\
&= m^2M^2(u_1^2 + 2v_iu_i + v_i^2) \\
&= m^2M^2(u_1 + v_1)^2
\end{aligned}$$

[For those of you wishing to study Physics at University, this is typical of calculations you will be doing in your first year. The algebra looks daunting but careful and persistent work will get you to the answer.]

Hence

$$v_f = \frac{m(Mu_1 - mv_1) \pm mM(u_1 + v_1)}{m(m+M)}$$

Taking the plus sign gives (the minus sign does not give a physical solution):

$$\begin{aligned}
v_f &= \frac{m(Mu_1 - mv_1) + mM(u_1 + v_1)}{m(m+M)} \\
&= \frac{2Mmu_1 + m(M-m)v_1}{m(m+M)} \\
&= \frac{2u_1 + (1 - \frac{m}{M})v_1}{1 + \frac{m}{M}}
\end{aligned}$$

The ratio $\frac{m}{M}$ (mass of satellite to mass of planet) is miniscule so we can ignore it. Then

$$v_f \approx 2u_1 + v_1$$

The satellite leaves the planet with a speed that increased by twice the speed of the planet! A bit of extra algebra shows that

$$u_2 = \frac{u_1 - \frac{m}{M}(u_1 + 2v_i)}{1 + \frac{m}{M}}$$

which in the limit of small $\frac{m}{M}$ becomes

$$u_2 \approx u_1 - 2(u_1 + v_i)\frac{m}{M} \approx u_1$$

That is to say, in the approximation of negligible satellite mass compared to that of the planet the speed of the planet stays the same. Of course, in reality, the speed of the planet is reduced by a tiny amount which accounts for the increased kinetic energy of the satellite. This works because the planet is moving. If the planet were stationary, the satellite would accelerate as it approached the planet and then decelerate as it left. The speed far away would be the same as the initial speed.

The kinetic energy gained by the satellite is

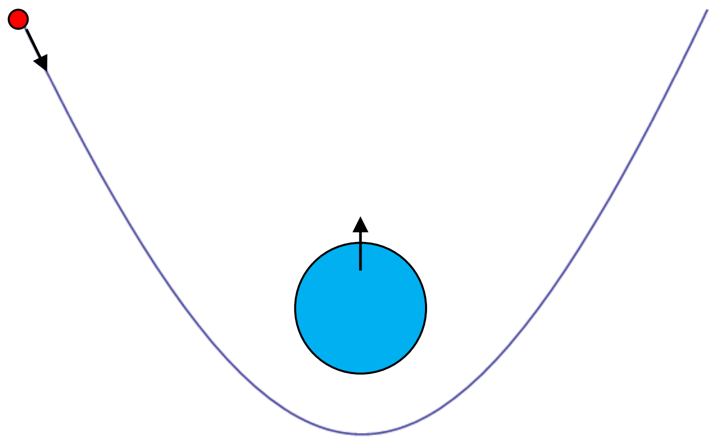
$$\Delta E = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \approx \frac{1}{2}m(2u_1 + v_1)^2 - \frac{1}{2}mv_1^2 = 2m(u_1^2 + 2u_1v_1)$$

The change in kinetic energy of the planet is

$$\begin{aligned}
\Delta E &= \frac{1}{2}Mu_2^2 - \frac{1}{2}Mu_1^2 \\
&= \frac{1}{2}M(u_1 - 2(u_1 + v_1)\frac{m}{M})^2 - \frac{1}{2}Mu_1^2 \\
&\approx \frac{1}{2}Mu_1^2 - \frac{1}{2} \times 4 \times M \times (u_1^2 + 2u_1v_1)\frac{m}{M}) - \frac{1}{2}Mu_1^2 \text{ (neglect terms in } (\frac{m}{M})^2) \\
&= -2m(u_1^2 + 2u_1v_1)
\end{aligned}$$

in precise agreement to what was gained by the satellite.

You can obtain similar results for more realistic orbits such as the one in the following figure.



The effect may also be used to *slow down* a satellite. It is left as research project for you to figure out how that is done.

**An aside:** consider a collision in which a body X of mass $m$ moving with speed $v$ collides with a stationary object Y of mass $M$. If the collision is elastic the methods used above may be used to find the velocities of the bodies after the collision. The results are

For X: $v_X = \dfrac{m-M}{m+M}v$

For Y: $v_Y = \dfrac{2m}{m+M}v$

Now, the *relative velocity of X with respect to Y* before the collision is $v-0=v$. After the collision it is $v_X - v_Y = \dfrac{m-M}{m+M}v - \dfrac{2m}{m+M}v = \dfrac{-m-M}{m+M} = -v$. We see the interesting result that in an elastic collision the relative velocities before and after the collision are equal in magnitude and opposite in direction. This is true even if both bodies are moving before the collision. (This is a powerful tool in quickly determining whether a collision is elastic or not.)

If we apply this result to our original problem here, the satellite approaching the moving planet,

we find right away that (we are assuming that the planet's speed does not change appreciably)

$$v_i - (-u) = -(-v_f - (-u))$$

giving

$$v_f = v_i + 2u$$